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On drift, diffusion and geometry

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Abstract

We present some reflections on the links between drift, diffusion and geometry. For this purpose, we examine different sources of "diffusion models", in physics and in mathematics. We observe that diffusion processes may arise from original models either deterministic, or random but where dynamics and noise are clearly delineated. In the end, we get a diffusion process where noise and dynamics ("drift") are generally intimately entangled in a second-order partial differential operator. We focus on the following questions. Are there implicit geometric structures to properly define a diffusion? How are drift/dynamics and diffusion mixed? Are there geometric structures needed to separate drift and diffusion? We stress the importance of recurrent differential geometric structures – connections and Riemannian metrics – needed to properly define a "diffusion term" and also to separate drift from diffusion.

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1. Introduction

Consider a set of particles in movement. One may say that drift is a general trend followed by these particles, while diffusion is random wandering. Going beyond words, there are different mathematical objects to capture the ideas of drift and diffusion.

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Concerning diffusion, Laplacian Δ and Brownian motion are intimately related. The Brownian motion may be seen as the prototype of continuous time noise. This diffusion process shares, with its infinitesimal generator $\frac{1}{2}\Delta$, strong spatial symmetries as the invariance by the Euclidian group. It is a sort of "pure noise" on $\mathbb{R}^n$, in the sense that no direction is privileged (isotropy, absence of drift). The drift is generally a vector field, giving direction to the solutions of an ordinary differential equation or driving the partial differential equation (PDE) equation for the density.

We shall make a brief review of different sources of "diffusion models". We shall focus on the following questions:

(1) are there implicit geometric structures to properly define a diffusion?
(2) how are drift and diffusion mixed?
(3) are there geometric structures needed to separate drift and diffusion?

We shall see that diffusion processes may arise from original models either deterministic, or random but where dynamics and noise are clearly delineated. In the end, we get a diffusion process where noise and dynamics ("drift") are generally intimately entangled, either in a second-order partial differential operator, or in a stochastic differential equation. We shall question this linkage and examine the geometric structures that may be needed to properly define a "diffusion term" and also to separate drift from diffusion. We shall also observe recurrent differential geometric structures: connections, Riemannian metrics.

This paper originates from reflections following a series of papers [1–5]. Classically associating a Riemannian metric g to a nondegenerate elliptic operator L on a manifold $\mathbb{M}$ [6], we exploited geometric properties of the Riemannian manifold $(\mathbb{M}, \mathsf{g})$ to exhibit properties of the diffusion process with infinitesimal generator L, and study various types of problem (symmetries, finite dimensional filters, group invariant solutions). Starting from stochastic problems formulated in terms of diffusion and drift, we were thus led to geometric problems: therefore, our reflection focused on the link between drift/diffusion and geometry. This paper is a tentative review to clarify the question.

Section 2 collects the mathematical background which will appear recurrently. In Sections 3 and 4, we revisit two physical models of diffusion – Fick's law and deterministic interacting particles systems – and we focus on the role of underlying geometric structures. Then, in Section 5, we examine diffusion processes and stochastic differential equations under the same geometrical angle. After having seen examples of how noise and dynamics entangle and the role of geometric structures, we turn to the reverse problem in Section 6. We shall examine operations to disentangle noise and dynamics, and try to properly define the "drift" of a diffusion process. In conclusion, we sum up our observations and sketch some recurrent facts.

2. Mathematical background

We collect here the main mathematical background needed in the sequel. In what follows, $\mathbb{M}$ is a smooth manifold.

2.1. Connection on a manifold (of dimension n)

A *connection* on a manifold $\mathbb{M}$ is any of the three following objects:

(1) A linear mapping Hess from smooth functions on $\mathbb{M}$ to symmetric bilinear forms on $\mathbb{M}$, such that for all smooth function φ [12, p. 32]:

$$\mathrm{Hess}(\varphi^2) = 2\varphi\,\mathrm{Hess}(\varphi) + 2\,\mathrm{d}\varphi \otimes 2\,\mathrm{d}\varphi. \tag{1}$$

(2) A C^∞-linear mapping F from $\tau_2\mathbb{M}$ – the space of *tangent vectors of order 2* (smooth fields of second-order differential operators L, with no zero order term) – to $\tau_1\mathbb{M}$ – the space of vector fields, that is tangent vectors of order 1 (smooth fields of first-order differential operators L, with no zero order term) – such that [12, p. 105]:

$$F : \tau_2\mathbb{M} \to \tau_1\mathbb{M}, \qquad F(f) = f \quad \text{for } f \in \tau_1\mathbb{M}. \tag{2}$$

(3) A covariant derivative D on $\mathbb{M}$, that is a linear mapping

$$\tau_1\mathbb{M} \times \tau_1\mathbb{M} \to \tau_1\mathbb{M}, \quad (f, g) \mapsto D_f g \tag{3}$$

which is punctual in the first argument and a derivative in the second argument.

Equivalence between these definitions comes from the following formulas [12, p. 35], where f and g are vector fields on $\mathbb{M}$, φ is a smooth function on $\mathbb{M}$:

$$\mathrm{Hess}\,\varphi(f, g) = fg\varphi - (D_f g)\varphi \quad \text{and} \quad D_f g = F(fg). \tag{4}$$

In the above equations, fg is a second-order differential operator with no zero order term, and $D_f g$ is a vector field.

To emphasize that a vector field f may be seen as a first-order differential operator with no zero order term, we shall often use the notation $\mathcal{L}_f$ for the *Lie differential along the vector field* f. In coordinates, if

$$f = \sum_{i=1}^{n} f^i(x)\frac{\partial}{\partial x_i} \tag{5}$$

and φ is a smooth function on $\mathbb{M}$, we have, for all $x \in \mathbb{M}$:

$$(\mathcal{L}_f\varphi)(x) = (f\varphi)(x) = \langle f, d\varphi\rangle(x) = \sum_{i=1}^{n} f^i(x)\frac{\partial\varphi}{\partial x_i}(x). \tag{6}$$

With obvious notations, we have

$$(\mathcal{L}_f\mathcal{L}_g\varphi)(x) = (fg\varphi)(x) = \sum_{i,j} g^j(x) f^i(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x) + \sum_{i,j} f^i(x)\frac{\partial g^j}{\partial x_i}(x)\frac{\partial\varphi}{\partial x_j}(x).$$

2.2. Riemannian manifold

Let us assume that $\mathbb{M}$ carries a Riemannian metric g: $(\mathbb{M}, \mathsf{g})$ is a Riemannian manifold.

2.2.1. Codifferential

The Hodge duality star-operator $*$ maps k-forms on $(n-k)$-forms (see [7, p. 457]).

For instance, when $\mathsf{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is the Euclidian flat metrics on $\mathbb{R}^3$, we have $*\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z$, $*\mathrm{d}y = -\mathrm{d}x \wedge \mathrm{d}z$, $*\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y$, and $*(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x$, $*(\mathrm{d}x \wedge \mathrm{d}z) = -\mathrm{d}y$, $*(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z$.

The *codifferential* δ is obtained from the Hodge duality star-operator $*$ and, when restricted to n-forms, is given by the formula [7, p. 457]:

$$\delta = - * \mathrm{d} * . \tag{7}$$

When $\mathsf{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ and $\alpha = a\,\mathrm{d}y \wedge \mathrm{d}z - b\,\mathrm{d}x \wedge \mathrm{d}z + c\,\mathrm{d}x \wedge \mathrm{d}y$, then $\delta\alpha = (c_y - b_z)dx + (a_z - c_x)dy + (b_x - a_y)\,\mathrm{d}z$, where $c_y = \frac{\partial c}{\partial y}$, etc.

2.2.2. Laplace–Beltrami operator

The *Laplace–Beltrami operator* is given by

$$\Delta_{\mathsf{g}} = \mathrm{div}_{\mathsf{g}} \nabla_{\mathsf{g}} = *\mathrm{d} * \mathrm{d}. \tag{8}$$

When $\mathsf{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, $\Delta_{\mathsf{g}} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

2.2.3. Levi–Civita connection

There exists a unique connection D such that $D\mathsf{g} = 0$: this is the so called *Levi–Civita connection* [19]. The associated C^∞-linear mapping F_{g} from $\tau_2\mathbb{M}$ to $\tau_1\mathbb{M}$ satisfies $F_{\mathsf{g}}(\Delta_{\mathsf{g}}) = 0$ [12, p. 105].

When $\mathsf{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, $D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial z} = 0$, and the same for all other coordinates (the Levi–Civita connection is flat).

2.3. Diffusion processes on a smooth manifold

We briefly recall here the definition of a diffusion on a smooth manifold and of its infinitesimal generator [11, p. 202]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, satisfying the usual conditions (when the filtration is not specified, it is that generated by the diffusion). Let also L be an elliptic partial differential operator on the manifold $\mathbb{M}$, written in a given coordinate system $x_1, \ldots, x_n$ as

$$L = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i}. \tag{9}$$

Let $\mathbb{M}' = \mathbb{M} \cup \{\vartheta\}$, where ϑ is a terminal point. By convention, any smooth φ on $\mathbb{M}$ extends to $\mathbb{M}'$ by $\varphi(\vartheta) = \vartheta$ and any transformation ϕ from $\mathbb{M}$ to $\mathbb{M}$ extends from $\mathbb{M}'$ to $\mathbb{M}'$ by $\phi(\vartheta) = \vartheta$. Let (ξ) be a family $(\xi_x)_{x\in\mathbb{M}}$ of $\mathbb{M}'$-valued, $\mathcal{F}$-adapted stochastic processes such that

(1) a.s., $\xi_x(0) = x$,
(2) a.s., there exists $\zeta(\omega) \in [0, +\infty]$ such that

(a) $t \in [0, \zeta(\omega)) \mapsto \xi_x(t)$ is continuous,
(b) $\xi_x(t) = \vartheta$ for $t \geq \zeta$,
(3) for all continuous bounded function φ on $\mathbb{M}'$,

$$M^{\varphi}(t) = \varphi(\xi_x(t)) - \varphi(\xi_x(0)) - \int_0^t (L\varphi)(\xi_x(s))\,\mathrm{d}s$$

is a martingale.

The process (ξ_x) is said to be a *diffusion with infinitesimal generator L and starting at x*.

When $\mathbb{M} = \mathbb{R}^n$, the n-dimensional *Brownian motion* is a diffusion with infinitesimal generator $\frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

2.4. Stochastic differential equations on a manifold

Stochastic differential equations (SDE) give rise to diffusion processes. For the geometrical aspects, our main reference here is Emery's book [12], and we shall follow its notations.

Let $T_y\mathbb{M}$ denote the tangent space at $y \in \mathbb{M}$, $\tau_1\mathbb{M}$ the space of vector fields on $\mathbb{M}$ (tangent vectors of order 1), and $\tau_1^{\star}\mathbb{M}$ the space of 1-forms on $\mathbb{M}$. Let $g_0, g_1, \ldots, g_m$ be vector fields on $\mathbb{M}$, and $(B_t^1, \ldots, B_t^m)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^m$.

In the sequel, when we write $R_t = S_t$, where $(R_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are stochastic processes, this means $\mathbb{P}(\forall t \geq 0, R_t = S_t) = 1$.

2.4.1. Stratonovich stochastic differential equations

Defining a Stratonovich SDE on a manifold does not require any particular geometric structure. One starts from the definition of the Stratonovich integral of semi-martingales with respect to a semi-martingale, then gives meaning to the Stratonovich SDE

$$\mathrm{d}Y_t = g_0(Y_t)\,\mathrm{d}t + \sum_{l=1}^m g_l(Y_t) \circ \mathrm{d}B_t^l \tag{10}$$

as follows. A process (Y_t) on $\mathbb{M}$ satisfies the Stratonovich SDE (10) if and only if, for all smooth function φ with compact support on $\mathbb{M}$, we have [11, p. 248]:

$$\varphi(Y_t) - \varphi(Y_0) = \int_0^t \mathcal{L}_{g_0}\varphi(Y_s)\,\mathrm{d}s + \sum_{l=1}^m \int_0^t \mathcal{L}_{g_l}\varphi(Y_s) \circ \mathrm{d}B_s^l, \tag{11}$$

where the last integral is a Stratonovich integral.

To the SDE (10), we may associate a diffusion process with infinitesimal generator

$$L = \mathcal{L}_f + \frac{1}{2}\sum_{j=1}^m \mathcal{L}_{g_j}^2 \tag{12}$$

that is

$$\forall\, \varphi \in C^{\infty}(\mathbb{M}), \quad L\varphi = \mathcal{L}_f\varphi + \frac{1}{2}\sum_{j=1}^{m}\mathcal{L}_{g_j}(\mathcal{L}_{g_j}\varphi). \tag{13}$$

2.4.2. *Itô stochastic differential equations*

To properly define an Itô stochastic differential equation, Emery claims that one needs an additionnal geometric structure known as connection. Indeed, *without connection, you cannot define a martingale on a manifold*.

Emery defines the Itô integral $\int\langle\alpha, F\,\mathrm{d}Y\rangle$ of a 1-form α along a semi-martingale Y and the integral $\int b(\mathrm{d}Y, \mathrm{d}Y)$ of a bilinear form b [12, Chapter VI]. For the case when $\alpha = \mathrm{d}\varphi$ and $b = \operatorname{Hess}\varphi$ (φ is a smooth function), this allows to give meaning to the following Itô SDE :

$$\mathrm{d}Y_t = g_0(Y_t)\,\mathrm{d}t + \sum_{l=1}^{m} g_l(Y_t)\,\mathrm{d}B_t^l. \tag{14}$$

One says that a process $(Y_t)_{t\geq 0}$ on $\mathbb{M}$ satisfies the Itô SDE (14) if and only if, for all smooth function φ with compact support on $\mathbb{M}$, we have [12, p. 109]:

$$\varphi(Y_t) - \varphi(Y_0) - \frac{1}{2}\int_0^t \operatorname{Hess}\varphi(\mathrm{d}Y_s, \mathrm{d}Y_s)$$
$$= \int_0^t \mathcal{L}_{g_0}\varphi(Y_s)\,\mathrm{d}s + \sum_{l=1}^{m}\int_0^t \mathcal{L}_{g_l}\varphi(Y_s)\,\mathrm{d}B_s^l. \tag{15}$$

3. Geometric structures in Fick's law of diffusion

Consider a moving fluid in a domain of $\mathbb{R}^3$, described by its concentration $c(x) \in \mathbb{R}_+$ and speed $v(x) \in \mathbb{R}^3$ at each point x of the domain (we follow [7]). The mass balance equation gives the following first-order PDE:

$$\frac{\partial c}{\partial t} + \operatorname{div}(cv) = 0. \tag{16}$$

The speed v is a vector field representing the transport dynamics.

Diffusion is usually modeled as an additional term

$$\text{diffusion term} = -k\nabla c \quad (k > 0). \tag{17}$$

This is *Fick's law*. The mass balance equation now gives the following parabolic PDE:

$$\frac{\partial c}{\partial t} + \operatorname{div}(cv - k\nabla c) = 0. \tag{18}$$

Since $\operatorname{div}\nabla = \Delta$, the Laplacian on $\mathbb{R}^3$, diffusion is associated with a parabolic PDE:

$$\frac{\partial c}{\partial t} + \operatorname{div}(cv) - k\,\Delta c = 0. \tag{19}$$

3.1. Implicit geometric structures in Fick's law

We examine here the geometric structures adapted to the description of an advection phenomenon, then of a diffusion phenomenon.

In the model above, mention is implicitly made of two geometric objects:

(1) a differential 3-form or volume form (the volume form $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ on $\mathbb{R}^3$) that allows to define the concentration c as a density; to this volume form is directly associated a divergence operator ($\mathrm{div} = \partial/\partial x + \partial/\partial y + \partial/\partial z$ on $\mathbb{R}^3$);
(2) a Riemannian metric (the flat metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$) that allows to define the gradient ∇, and thus to linearly associate a vector field (∇c) to a function (c) as in Fick's law (17).

Thus, to describe the transport phenomenon (dynamics), one only needs a volume form. But, the representation of the diffusion phenomenon requires a richer structure, here a metric (and its associated volume form): indeed, *Fick's law – by its expression in $-k\nabla c$ – implicitly refers to a Riemannian metric* allowing to define a gradient.

3.2. Diffusion may be represented by a $(n-1)$-form

We shall now extend the above remarks in a more general analysis.

Let be given a smooth manifold $\mathbb{M}$ and a vector field v on $\mathbb{M}$, which characterize the dynamics of a system ($\dot{x} = v(x)$). Suppose also that $\mathbb{M}$ is orientable and supports a volume form μ, that allows to define densities.

If W is a smooth sub-domain of $\mathbb{M}$, we assume that the fluid mass contained in W at time t is $\int_W c_t\mu$, where $c(t,x) = c_t(x)$ is a smooth positive function on $[0,+\infty[\times\mathbb{M}$ (the concentration c_t is a mass density). The n-form $c_t\mu$ may thus be seen as a mass distribution on the smooth manifold $\mathbb{M}$: *$c_t\mu$ carries an intrinsic physical meaning, more than c_t* which depends of the choice of a reference volume form μ.

Let $(\phi_t)_{t\geq 0}$ denote the flow associated to the dynamics v. The displacement of the volume W into $\phi_t(W)$ and mass conservation are expressed via the volume integrals $\int_W c_0\mu$ and $\int_{\phi_t(W)} c_t\mu$. On the other hand, the *mass exchange by diffusion* is represented by a *surface integral* $\int_0^t \mathrm{d}s \int_{\partial\phi_s(W)} \alpha$. The term α is a $(n-1)$-form which models the diffusion phenomenon. Mass conservation then gives

$$\left(\frac{\partial c_t}{\partial t} + \mathrm{div}(c_t v)\right)\mu = \mathrm{d}\alpha, \tag{20}$$

where we have used Stokes' theorem $\int_{\partial W}\alpha = \int_W \mathrm{d}\alpha$. Thus, to model diffusion and "close" Eq. (20) with respect to c_t, we need to express the $(n-1)$-form α as a function of c_t. This "closure" may be done in a linear way as follows.

3.3. A proposal of linear diffusion model requiring a Riemannian metric

By a linear diffusion model, we mean an intrinsic way of linearly associating a $(n-1)$-form α (or an exact n-form $\mathrm{d}\alpha$) to the n-form $c\mu$ (intrinsic mass distribution). Now, this operation is far from being natural without additional structure.

Indeed, the exterior differential operator d on exterior forms allows to “go up” from 0-forms to 1-forms, from 1-forms to 2-forms,..., from $(n-1)$-forms to n-forms. But, without additional geometric structure, there is no natural “descending” operator from n-forms to $(n-1)$-forms.

However, if the smooth manifold $\mathbb{M}$ is equipped with a Riemannian metric g, there exists a “descending” operator δ, the exterior codifferential which maps n-forms to $(n-1)$-forms, $(n-1)$-forms to $(n-2)$-forms, ..., 1-forms to 0-forms.

Let us now assume that $\mathbb{M}$ carries a Riemannian metric g, and show what a linear diffusion model may be, by making use of the exterior codifferential δ introduced in Section 2.2. If c is a smooth nonnegative function, representing concentration with respect to the Riemannian volume form (up to a multiplicative constant), we propose to model the diffusion phenomenon by the following $(n-1)$-form α:

$$\text{“diffusion”} = \alpha = -\delta(c\Omega_{\mathsf{g}}). \tag{21}$$

An easy computation [7, p. 427] shows that

$$\mathrm{d}\alpha = -\mathrm{d}\delta(c\Omega_{\mathsf{g}}) = \mathrm{d} * \mathrm{d} * (c\Omega_{\mathsf{g}}) = \mathrm{d} * \mathrm{d}c = (\Delta_{\mathsf{g}} c)\Omega_{\mathsf{g}} \tag{22}$$

since $*\Omega_{\mathsf{g}} = 1$ and where $\Delta_{\mathsf{g}} = \mathrm{div}_{\mathsf{g}} \nabla_{\mathsf{g}} = *\mathrm{d} * \mathrm{d}$ is the Laplace–Beltrami operator. Replacing this latter expression in (20) with $\mu = \Omega_{\mathsf{g}}$, we get a parabolic PDE satisfied by the concentration c_t:

$$\frac{\partial c_t}{\partial t} + \mathrm{div}_{\mathsf{g}}(c_t v) = \Delta_{\mathsf{g}} c_t. \tag{23}$$

3.4. How we can recover Fick’s law

Eq. (23) hereabove may be written as

$$\frac{\partial c_t}{\partial t} + \mathrm{div}_{\mathsf{g}}(c_t v - \nabla_{\mathsf{g}} c_t) = 0. \tag{24}$$

One thus recognizes a more traditional interpretation of diffusion as gradient of the concentration c. This appears also when we write the diffusion flux through a surface by [7, p. 484]:

$$\int_{\partial W} \alpha = \int_W \mathrm{d}\alpha = \int_W (\Delta_{\mathsf{g}} c)\Omega_{\mathsf{g}} = \int_W (\mathrm{div}_{\mathsf{g}} \nabla_{\mathsf{g}} c)\Omega_{\mathsf{g}} = \int_{\partial W} \mathsf{g}(\nabla_{\mathsf{g}} c, \vec{n})\, \mathrm{d}\sigma, \tag{25}$$

where $\vec{n}$ is the exterior normal on ∂W and σ the surface measure on ∂W, and where we used Stokes’ theorem.

3.5. Comments on drift, diffusion and underlying geometric structures

Thus, the mathematical representation of a diffusion phenomenon requires geometric structures, like Riemannian metrics: if diffusion is seen as a surface term (a $(n-1)$-form) while a mass distribution is an n-form, a diffusion model “descends” from n-forms towards $(n-1)$-forms; however, the exterior derivation d goes the other way round, and dualizing d requires an additional structure such as a Riemannian metric.

As to drift v and diffusion $-k\nabla c$, they appear distinctly in Eq. (19). However, this decomposition is made possible by the flat structure of $\mathbb{R}^3$. Drift v and diffusion $-\nabla_{\mathfrak{g}} c$ also appear in (24), and depend upon $\mathfrak{g}$. Thus, such a decomposition between drift and diffusion depends upon a Riemannian metric.

4. Geometric structures in deterministic interacting particles systems

Deterministic interacting particles systems are a source of diffusion models: limit equations for density often are parabolic PDE s.

Our references here are the works of Spohn [8] (see also [9]).

Deterministic particles interacting in a potential are represented by a Hamiltonian

$$H = \sum_j \frac{1}{2m_j} p_j^2 + \sum_{i<j} V_{ij}(q_i - q_j), \tag{26}$$

where $(q_j, p_j) \in \mathbb{R}^d \times \mathbb{R}^d$ is the couple position-momentum of particle j with mass m_j, and V_{ij} is a central potential ($d = 3$ in physical applications).

According to the way a parameter ε appears in the Hamiltonian, one obtains different 'Markovian limit models' for $\varepsilon \downarrow 0$, that is parabolic PDE s satisfied by the probability density $\pi(q, p, t)$ of finding a given particle at point (q, p) at time t.

4.1. Lorentz gas

The Lorentz gas [8, p. 572] is a particle of mass M moving through infinitely heavy randomly distributed scatterers $q_1, q_2, \ldots$ (with uniform density ρ, where $\rho \in \mathbb{R}_+^*$). The formal Hamiltonian is

$$H = \frac{1}{2M} p^2 + \sum_j V(q - q_j). \tag{27}$$

To the weak coupling limit [8, p. 574], characterized by the scaling

$$V_\varepsilon(q) = \varepsilon^{1/2} V(q/\varepsilon) \quad \text{and} \quad \rho_\varepsilon = \varepsilon^{-d}\rho, \tag{28}$$

one gets a *diffusion resulting from numerous, but weak, collisions*.

The probability density $\pi(q, p, t)$ of finding the Lorentz particle at point (q, p) at time t is given by the linear Landau equation [8, p. 575]:

$$\frac{\partial}{\partial t}\pi(q, p, t) = \left(-p\nabla_q + \rho \sum_{i,j} \frac{\partial}{\partial p_i} D_{ij}(p) \frac{\partial}{\partial p_j} \right) \pi(q, p, t). \tag{29}$$

There exists a constant α (depending only on V) such that the "diffusion matrix" is given by:

$$D_{ij}(p) = \frac{\alpha}{2|p|} \left(\delta_{ij} - \frac{p_i p_j}{|p|^2} \right). \tag{30}$$

One may check that the symmetric matrix with general term $D_{ij}(p)$ is degenerate since p belongs to the kernel (in spherical coordinates, the diffusion term is $\frac{\rho\alpha}{2|p|}\Delta_{|p|}$, where $\Delta_{|p|}$ is the Laplace–Beltrami operator on the sphere with radius $|p|$). The diffusion term is thus degenerate and may not be deduced from a Riemannian metric on the space of momentum.

4.2. Rayleigh's gas

The Rayleigh's gas [8, p. 579] is a particle of mass M moving through a fluid consisting of infinitely heavy randomly distributed scatterers $q_1, q_2, \ldots$ (with uniform density ρ, where $\rho \in \mathbb{R}_+^*$) which are affected by collisions with the Rayleigh particle. The formal Hamiltonian is

$$H = \frac{1}{2M}p^2 + \sum_j V(q - q_j) + \sum_j \frac{1}{2m}p_j^2 + \sum_{i<j} U(q_i - q_j). \tag{31}$$

To the Brownian motion limit [8, p. 580], characterized by the scaling

$$V_\varepsilon(q) = V(q/\varepsilon), \qquad \rho_\varepsilon = \varepsilon^{-d}\rho, \qquad m_\varepsilon = \varepsilon^2 m, \qquad p_{j,\varepsilon} = \varepsilon p_j \tag{32}$$

one expects the probability density $\pi(q, p, t)$ of finding the Rayleigh particle at point (q, p) at time t to satisfy the Fokker–Planck equation of an Ornstein–Uhlenbeck process

$$\frac{\partial}{\partial t}\pi(q, p, t) = \left(-\frac{1}{M}p\nabla_q + \frac{D\beta}{M}\nabla_p \cdot p + D\Delta_p\right)\pi(q, p, t), \tag{33}$$

in which D, the diffusive term, is a constant scalar (β is the inverse of the temperature of the fluid).

This conjecture is proved in a limited number of cases, in particular on the real line for an infinite system of hard balls with mass m and a hard ball of mass M, all of zero length and evolving by elastic collision [8, p. 584].

4.3. Hydrodynamic limit

Up to now, Markovian limits are obtained by letting a physical parameter (interaction force, fluid density, inverse of a mass) go to zero. They can also be obtained for long time limits:

$$\lim_{\varepsilon\downarrow 0} \varepsilon q\left(\frac{t}{\varepsilon^2}\right). \tag{34}$$

In [10, p. 171], one finds the first proper result of *convergence of a purely deterministic dynamic system towards a Brownian motion*: for a field of diffusing centers (hard spheres) periodic on a rectangle and with finite horizon (bounded mean free length), properly normalized trajectories converge in law towards a Brownian motion.

For Lorentz gas and Rayleigh gas (test particle or system of particles), there are proofs of convergence towards a Brownian motion in a limited number of cases.

4.4. Comments on drift, diffusion and underlying geometric structures

The models above all lead to parabolic PDE s like (29) and (33). The underlying particle evolves in the phase space $\mathbb{R}^d \times \mathbb{R}^d$ with kinetic energy $p^2/2m$: this corresponds to a classical description in which the particle travels along straight lines between shocks, that is along geodesics for the flat connection.

When, in addition, the density of scatterers is constant, the second-order term for the Lorentz gas is proportional to $1/|p|$ times the Laplace–Beltrami operator $\Delta_{|p|}$ on the sphere with radius $|p|$. We have already pointed out that this diffusion term is degenerate and may not be deduced from a Riemannian metric on the space of momentum. Other second-order terms than the usual Laplacian are possible when the density of scatterers is not constant [8, p. 575].

For the Rayleigh's gas with constant density of scatterers, the diffusion term is proportional to the usual Laplacian Δ_p, thus inherited from the flat Riemannian metric and related to the flat connection.

As to drift and diffusion, systems of interacting particles for which Markovian limits have been studied do not possess drift forces that would act on all particles, but only interactions between particles. Indeed, mathematically, the Hamiltonian is given by

$$H = \sum_j \frac{1}{2m_j} p_j^2 + \sum_{i<j} V_{ij}(q_i - q_j) \tag{35}$$

and not by

$$H = \sum_j \frac{1}{2m_j} p_j^2 + \sum_{i<j} V_{ij}(q_i - q_j) + \sum_j V(q_j), \tag{36}$$

where the last term would represent a force $-\nabla_q V$ which is not an interaction. Thus, with available mathematical results on Markovian limits, one cannot apprehend how the drift term $\sum_j V(q_j)$ and the interactions $\sum_{i<j} V_{ij}(q_i - q_j)$ would combine in a diffusion limit model.

5. Geometric aspects of mathematical diffusion processes and stochastic differential equations

We have briefly recalled in Section 2 what is a diffusion process in probability theory, and the links with stochastic differential equations. We here recall how diffusion processes may be obtained as limits of randomly perturbed differential systems. This is an opportunity to observe that the entangling of a "pure drift" with a "pure noise" is not a straightforward operation. It is mediated through a formalism which may require geometric structures, such as a connection or a Riemannian metric.

5.1. Randomly perturbed differential systems as sources of diffusion processes

Our references here are the works of Papanicolaou [15,16] and of Kesten and Papanicolaou [17,18] where it is shown that diffusion processes may be obtained as limit in law of certain sequences of randomly linearly perturbed ODEs (ordinary differential equations).

5.1.1. Randomly linearly perturbed ODE*s on* $\mathbb{R}^n$ *converge in law towards Stratonovich* SDE*s*

Let f be a vector field on $\mathbb{R}^n$, and let us model random perturbations of the ODE $\dot{x} = f(x)$ as

$$\frac{\mathrm{d}x^\varepsilon(t)}{\mathrm{d}t} = f(x^\varepsilon(t)) + \frac{1}{\varepsilon}\sum_{i=1}^{m} g_i(x^\varepsilon(t))U_I\left(\frac{t}{\varepsilon^2}\right), \tag{37}$$

where $\varepsilon > 0$ and

- $U_1(t), \ldots, U_m(t)$ are m Markov processes (either with continuous trajectories or with jumps), independent, stationary, with a unique invariant measure, with mean zero and standart deviation 1;
- $g_1, \ldots, g_m$ are m vector fields.

It is shown in [13, Chapter 10] that the sequence of processes $(x^\varepsilon(t))$ converges in law, when $\varepsilon \downarrow 0$, towards a diffusion with infinitesimal generator (12), that is to say converges in law towards the solution (x_t) of the following SDE (in Stratonovich sense, where $(B_t^1, \ldots, B_t^n)_{t\geq 0}$ is a Brownian motion):

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sum_{i=1}^{n} g_i(x_t) \circ \mathrm{d}B_t^i. \tag{38}$$

Thus SDE s in the Stratonovich sense appear as limit models of ODEs with random independent coefficients.

However, we shall see in what follows that this property is related to the linear way according to which random perturbations drive the ODE in (37) and to their statistical independence.

5.1.2. Randomly perturbed ODE*s on a manifold converge in law towards diffusion processes*

There exists an extension of the above result in the more general framework of a smooth manifold developed in [15, pp. 358–359].

Let be given a smooth manifold $\mathbb{M}$ and

- a Markov process $U(t)$ on a compact metric space S (either with continuous trajectories or with jumps), stationary, with a unique invariant measure Π,
- a family $f(\cdot, u)$, $u \in S$ of vector fields on $\mathbb{M}$,
- a family $g(\cdot, u)$, $u \in S$ of vector fields on $\mathbb{M}$.

Assume that $g(\cdot,\cdot)$ and Π are related by

$$\mathbb{E}_{\Pi}((g(\cdot, U(0)))) = \int_S g(\cdot, u)\, \mathrm{d}\Pi(u) \equiv 0$$

in the sense that, for all function $\varphi \in C^{\infty}(\mathbb{M})$ and for all $x \in \mathbb{M}$, one has

$$\mathbb{E}_{\Pi}(\mathcal{L}_{g(\cdot,U(0))}\varphi(x)) = \mathbb{E}_{\Pi}(\langle g(\cdot, U(0)), \mathrm{d}\varphi\rangle(x)) = \int_S \langle g(\cdot, u), \mathrm{d}\varphi\rangle(x)\, \mathrm{d}\Pi(u) = 0.$$

Under regularity assumptions on the dependence in $u \in S$, it is shown in [15] that the sequence of processes $(x^{\varepsilon}(t))$ defined as solutions of the following ODEs ($\varepsilon > 0$)

$$\frac{\mathrm{d}x^{\varepsilon}(t)}{\mathrm{d}t} = f\left(x^{\varepsilon}(t), U\left(\frac{t}{\varepsilon^2}\right)\right) + \frac{1}{\varepsilon} g\left(x^{\varepsilon}(t), U\left(\frac{t}{\varepsilon^2}\right)\right) \tag{39}$$

converge in law, when $\varepsilon \to 0$, towards a diffusion process on $\mathbb{M}$ with generator L. This latter is defined on any function $\varphi \in C^{\infty}(\mathbb{M})$ by

$$(L\varphi)(x) = \int_0^{+\infty} \mathrm{d}t\mathbb{E}_{\Pi}(\mathcal{L}_{g(\cdot,U(0))}\mathcal{L}_{g(\cdot,U(t))}\varphi(x)) + \mathbb{E}_{\Pi}(\mathcal{L}_{f(\cdot,U(0))}\varphi(x)). \tag{40}$$

5.2. Comments on drift, diffusion and underlying geometric structures

With what we have just seen above, it appears that the entangling of a “pure drift” with a “pure noise” is not a straightforward operation. It is mediated through a formalism which may require geometric structures.

To illustrate this, let us consider a vector field f on $\mathbb{R}^n$. Adding a Brownian motion $(B_t)_{t\geq 0}$ to the ODE

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{41}$$

gives a SDE

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \mathrm{d}B_t. \tag{42}$$

In this case, there is no ambiguity as to the meaning of $\mathrm{d}B_t$: Itô and Stratonovich integrals coincide here.

However, this “Brownian motion addition” is intimately made possible by the linear structure of $\mathbb{R}^n$. Should we forget the linear structure, then the term “$+\mathrm{d}B_t$” would loose sense. It might be replaced by

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sum_{j=1}^{m} g_j(x_t)\,\mathrm{d}B_t^j, \tag{43}$$

where $g_1, \ldots, g_m$ are vector fields on $\mathbb{R}^n$. But, how should we choose $g_1, \ldots, g_m$? And how should we interpret the above equation: with Itô or with Stratonovich integrals? With which connection in the Itô case?

More generally, one can see the question of incorporating noise in a ODE as the way of extending a first-order differential operator (or vector field) into a second-order one (or diffusion operator). Indeed, mathematically, things can be put this way.

On the one hand, a diffusion process on a manifold $\mathbb{M}$ with infinitesimal generator (9) is a family $(\xi_x)_{x\in\mathbb{M}}$ of Markov processes with continuous paths such that, for all smooth function φ with compact support on $\mathbb{M}$,

$$\varphi(\xi_x(t)) - \varphi(\xi_x(0)) - \int_0^t (L\varphi)(\xi_x(s))\,\mathrm{d}s = \text{martingale}. \tag{44}$$

On the other hand, a dynamical system on a manifold $\mathbb{M}$ with vector field f is a family of trajectories $t \hookrightarrow x(t)$ such that, for all smooth function φ with compact support on $\mathbb{M}$,

$$\varphi(x(t)) - \varphi(x(0)) - \int_0^t (\mathcal{L}_f\varphi)(x(s))\,\mathrm{d}s = 0. \tag{45}$$

Moving from (45) to (44) may be seen as moving from the first-order differential operator f, given by (5), to the second-order differential operator L, given by (9). This consists in particular in adding second-order terms to f. We shall see in the next section that such terms are intimately related to a Riemannian structure.

To extract from a diffusion process a “pure noise” and a “pure drift” may be seen as to move from L (second-order partial differential operator) to f (vector field seen here as a first-order partial differential operator). We shall see in the next section that such an operation at least requires a connection.

6. Disentangling noise and dynamics

After having seen, in the previous sections, examples of how noise and dynamics entangle, we now turn to the reverse problem.

Given a diffusion process, are there operations to disentangle noise and dynamics in it? If such operations exist, are they related to specific geometric structures? What exactly is the “drift” of a diffusion process ? In this section, we shall try and answer these questions.

6.1. The drift of a SDE *depends upon a connection*

On $\mathbb{R}^n$, the Itô–Stratonovich conversion formula for SDE s is well known [13, pp. 170–171] and makes use of derivatives of the coordinates of the vector fields $g_1, \ldots, g_m$.

On a general manifold $\mathbb{M}$, equipped with a connection so that Itô integrals may be defined, the Itô–Stratonovich conversion formula is given as follows [14].

Let $g_0, g_1, \ldots, g_m$ be vector fields on a smooth manifold $\mathbb{M}$, equipped with a covariant derivative D. The following two SDE s are equivalent:

$$\mathrm{d}Y_t = g_0(Y_t)\,\mathrm{d}t + \sum_{l=1}^{m} g_l(Y_t)\,\mathrm{d}B_t^l,$$

$$\mathrm{d}Y_t = \left[g_0(Y_t) - \frac{1}{2}\sum_{l=1}^{m} D_{g_l}g_l(Y_t)\right]\mathrm{d}t + \sum_{l=1}^{m} g_l(Y_t)\circ\mathrm{d}B_t^l. \tag{46}$$

The following two SDE s are equivalent:

$$\mathrm{d}Z_t = g_0(Z_t)\,\mathrm{d}t + \sum_{l=1}^{m} g_l(Z_t) \circ \mathrm{d}B_t^l,$$

$$\mathrm{d}Z_t = \left[g_0(Z_t) + \frac{1}{2} \sum_{l=1}^{m} D_{g_l} g_l(Z_t) \right] \mathrm{d}t + \sum_{l=1}^{m} g_l(Z_t)\,\mathrm{d}B_t^l. \tag{47}$$

Thus, the drift of a SDE depends upon the type of stochastic integral, Stratonovich or Itô, and upon a connection.

6.2. *You need a connection structure to extract a drift from a diffusion process*

The decomposition of a diffusion process as the sum of a drift term and of a "purely diffusive" term amounts to splitting a second-order differential operator into first- and second-order parts:

$$\frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} = \text{purely diffusive} + \text{drift}. \tag{48}$$

But, as is stated by Emery in [12, p. 76]: *second-order differential operator have no* (*intrinsic*) *first-order part*.

Now, as seen in Section 2.1, connections are in one-to-one correspondence with C^∞-linear mappings F from fields of second-order differential operators on a smooth manifold $\mathbb{M}$ towards those of order one, which satisfy $F(L) = L$ if L is of order one.

Thus, you need a connection to split a diffusion as the sum of a drift term and of a "purely diffusive" term : see [12, pp. 105–109], where this discussion is related to the impossibility of defining a martingale in a manifold without an additional structure [12, pp. 31–32].

6.3. *A drift and a Riemannian metric may be intrinsically associated to a nondegenerate diffusion operator*

Let L be a nondegenerate diffusion operator on a smooth manifold $\mathbb{M}$, given by (9). Assume that L is smooth (the functions $a^{ij}(\cdot)$, $b^i(\cdot)$ are smooth) and nondegenerate elliptic (the symmetric matrix $(a^{ij}(x))_{i,j=1,\ldots,n}$ is positive definite for all $x \in M$).

Thanks to this assumption, it is well known that we can introduce a Riemannian metric g on $\mathbb{M}$ as follows [6,11,20]. If $(a_{ij}(x))_{i,j=1,\ldots,n}$ denotes the inverse matrix of $(a^{ij}(x))_{i,j=1,\ldots,n}$, then

$$\mathsf{g} = \sum_{i,j=1}^{n} a_{ij}(x)\,\mathrm{d}x_i\,\mathrm{d}x_j \tag{49}$$

defines a Riemannian metric g on $\mathbb{M}$.

Moreover, if Δ_{g} is the Laplace–Beltrami operator (Laplacian) on the Riemannian manifold $(\mathbb{M}, \mathsf{g})$, there exists a unique smooth vector field f on $\mathbb{M}$ such that $\mathcal{L}$ can be written

$$L = \frac{1}{2}\Delta_{\mathsf{g}} + f. \tag{50}$$

Note that f depends not only on $b^1, \ldots, b^n$ in (9) but also on a^{ij}, i, $j = 1, \ldots, n$.

Thus, the vector field f is a natural candidate as drift term. It is also the image of L by the Levi–Civita connection F_{g} intrinsically associated to the Riemannian metric g (see Section 2.2). Indeed,

$$F_{\mathsf{g}}(L) = \frac{1}{2}F_{\mathsf{g}}(\Delta_{\mathsf{g}}) + F_{\mathsf{g}}(f) = f \tag{51}$$

since $F_{\mathsf{g}}(\Delta_{\mathsf{g}}) = 0$ and $F_{\mathsf{g}}(f) = f$, by (2) because f is a first-order partial differential operator (see Section 2.1). This establishes a link with the connection of the previous paragraph.

6.4. Laplacian, drift and Itô SDE

The above vector field f may also be considered as the drift of an SDE in Itô form. The proof of the following proposition is given in Appendix A.

Proposition 6.1. *Let* $(\mathbb{M}, \mathsf{g})$ *be a Riemannian manifold, admitting a* (*global*) *field of orthonormal frames, that is a family of vector fields* $g_1, \ldots, g_m$ *forming a basis such that* $\mathsf{g}(g_i, g_j) = \delta_{ij}$. *Let also f be a vector field on* $\mathbb{M}$.

Then, the following Itô SDE on the Riemannian manifold $(\mathbb{M}, \mathsf{g})$

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sum_{l=1}^{m} g_l(x_t)\,\mathrm{d}B_t^l \tag{52}$$

generates a diffusion process with infinitesimal generator

$$L = \frac{1}{2}\Delta_{\mathsf{g}} + \mathcal{L}_f. \tag{53}$$

Note that the existence of a family of vector fields $g_1, \ldots, g_n$ such that $\mathsf{g}(g_i, g_j) = \delta_{ij}$ and $n = \dim(\mathbb{M})$ may be excluded for topological reasons (as on the sphere S^2). In such a case, the above result would not hold.

6.5. You cannot associate unambiguously the drift of a Stratonovich SDE to a diffusion process

For a given diffusion operator L, there is no unique collection f, $g_1, \ldots, g_m$ of vector fields on $\mathbb{M}$ such that (12) holds true, that is, such that L is the infinitesimal generator of the diffusion process solution of the Stratonovich SDE (10).

6.6. *You generally loose the drift when randomly perturbed* ODE *s converge in law towards a diffusion process*

A vector field naturally emerges from the result of Section 5.1.2. Indeed, the following mapping which maps any function $\varphi \in C^{\infty}(\mathbb{M})$ towards the function

$$x \in \mathbb{M} \mapsto \mathbb{E}_{\Pi}(\langle f(\cdot, U(0)), \mathrm{d}\varphi\rangle(x)) = \mathbb{E}_{\Pi}(\mathcal{L}_{f(\cdot,U(0))}\varphi(x)) \tag{54}$$

has the characteristic properties of a vector field that we shall denote by $\mathbb{E}_{\Pi}(f(\cdot, U(0)))$.

It is clear that, if $f(\cdot, u)$ does not depend on $u \in S$ and is a simple vector field f, then $\mathbb{E}_{\Pi}(f(\cdot, U(0))) = f$, and f should be a natural candidate for drift of a diffusion process associated to L given by (40). However, things are not that simple.

Indeed, let us examine the second-order remaining part $L - \mathbb{E}_{\Pi}(f(\cdot, U(0)))$: it maps any function $\varphi \in C^{\infty}(\mathbb{M})$ to the function

$$x \in \mathbb{M} \mapsto \int_0^{+\infty} \mathrm{d}t\mathbb{E}_{\Pi}(\mathcal{L}_{g(\cdot,U(0))}\mathcal{L}_{g(\cdot,U(t))}\varphi(x)). \tag{55}$$

This mapping has the characteristic properties of a "second-order vector field", that is of a finite sum of terms AB and C, where A, B and C vector fields [12, pp. 75–76]. However, little more can be said of this term.

One can yet speak of the drift f in the specific case where the family $g(\cdot, u)$ is linear in $u \in \mathbb{R}^m$

$$g(\cdot, u) = \sum_{i=1}^{m} u_i g_i \tag{56}$$

and where the process $U(t)$ has independent components with mean zero and variance one. Indeed, the term $L - \mathbb{E}_{\Pi}(f(\cdot, U(0)))$ is equal to $\frac{1}{2}\sum_{j=1}^{m} \mathcal{L}_{g_j}^2$. We have already noticed this property in Section 5.1.1.

Otherwise, the mixing made by noise does not allow to extract privileged directions $g_1, \dots, g_m$.

7. Conclusion

So far, after having studied Itô and Stratonovich SDE s on manifolds, revisited different models which lead to diffusion processes and examined operations to disentangle noise and dynamics, we have encountered recurrent differential geometric structures: connections, Riemannian metrics.

Indeed, defining an Itô SDE on a manifold requires a connection, which appears as a covariant derivative in a general Itô–Stratonovich conversion formula for SDE s.

Indeed, the mathematical representation of a diffusion phenomenon requires geometric structures, like Riemannian metrics: if diffusion is seen as a surface term (an $(n-1)$-form) while a mass distribution is an n-form, a diffusion model "descends" from n-forms towards $(n-1)$-forms; however, the exterior derivation d goes the other way round, and dualizing d requires an additional structure such as a Riemannian metric.

Indeed, deterministic interacting particles systems like the Rayleigh gas exhibit Laplace–Beltrami operators derived from Riemannian metrics.

Indeed, splitting a diffusion into a drift term with finite variation and a purely diffusive term requires a connection (as the Levi–Civita connection associated to a Riemannian metric).

Although our trip in various domains – Fick's law, deterministic interacting particles systems, diffusion processes and stochastic differential equations – does not lead to a unified treatment (the question of drift for deterministic interacting particles systems is not treated in Section 6 for instance), it is however a source of various examples and arguments that point towards the need of additional structures (connection, Riemannian metric) to model noise entangled with continuous time dynamics.

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Appendix A. Proof of Proposition 6.1

Proof. By (46), the Itô SDE (52) may also be written as

$$\mathrm{d}x_t = \left(f - \frac{1}{2}\sum_{l=1}^{m} D_{g_l} g_l \right)(x_t)\,\mathrm{d}t + \sum_{l=1}^{m} g_l(x_t) \circ \mathrm{d}B_t^l$$

and its infinitesimal generator L is therefore, by Section 6.1,

$$L = \frac{1}{2}\sum_{l=1}^{m} \mathcal{L}_{g_l}^2 + \mathcal{L}_f - \frac{1}{2}\sum_{l=1}^{m} \mathcal{L}_{D_{g_l} g_l}. \tag{A.1}$$

We have for all smooth function φ

$$\nabla_{\mathsf{g}}\varphi = \begin{cases} \sum_{l=1}^{m} \mathsf{g}(\nabla_{\mathsf{g}}\varphi, g_l) g_l \text{ since } g_1, \ldots, g_k \text{ are orthonormal} \\ \sum_{l=1}^{m} (\mathcal{L}_{g_l}\varphi) g_l \text{ since } \mathsf{g}(\nabla_{\mathsf{g}}\varphi, g_l) = \langle d\varphi, g_l \rangle = \mathcal{L}_{g_l}\varphi. \end{cases}$$

Therefore, since

$$\mathrm{div}_{\mathsf{g}}([X, Y]) = X \cdot (\mathrm{div}_{\mathsf{g}}\, Y) - Y \cdot (\mathrm{div}_{\mathsf{g}}\, X) = \mathcal{L}_X(\mathrm{div}_{\mathsf{g}}\, Y) - \mathcal{L}_Y(\mathrm{div}_{\mathsf{g}}\, X)$$

for any vector fields X, Y, we have

$$\Delta_{\mathsf{g}}\varphi = \mathrm{div}_{\mathsf{g}}(\nabla_{\mathsf{g}}\varphi) = \sum_{l=1}^{m} \mathcal{L}_{g_l}^2 \varphi + \sum_{l=1}^{m} (\mathrm{div}_{\mathsf{g}}\, g_l)\mathcal{L}_{g_l}\varphi.$$

Thus, L given by (A.1) becomes

$$L = \frac{1}{2}\Delta_{\mathsf{g}} + f - \frac{1}{2}\left(\sum_{l=1}^{m} D_{g_l} g_l + \sum_{l=1}^{m}(\mathrm{div}_{\mathsf{g}}\, g_l) g_l\right).$$

There remains to prove that the last term is zero, that is

$$\sum_{l=1}^{m}(\mathrm{div}_{\mathsf{g}}\, g_l) g_l = -\sum_{l=1}^{m} D_{g_l} g_l,$$

and this amounts to proving that the coefficients of $-\sum_{l=1}^{m} D_{g_l} g_l$ in the orthonormal frame $g_1,\ldots, g_m$ are $\mathrm{div}_{\mathsf{g}}\, g_1,\ldots, \mathrm{div}_{\mathsf{g}}\, g_m$.

Now, since $\mathsf{g}(g_l, g_k) = \delta_{lk}$, we obtain, by appling D_{g_k} to this equality,

$$\mathsf{g}(D_{g_k} g_l, g_k) + \mathsf{g}(g_l, D_{g_k} g_k) = 0, \tag{A.2}$$

since the Levi–Civita connection D satisfies $D\mathsf{g} = 0$. We thus get, with classical differential geometry formulas [19]:

$$\mathrm{div}_{\mathsf{g}}\, g_l = \begin{cases} -\mathrm{tr}(A_{g_l}) \\ \sum_{k=1}^{m} \mathsf{g}(D_{g_k} g_l, g_k) \text{ since } A_{g_l} g_k = D_{g_k} g_l \\ -\sum_{k=1}^{m} \mathsf{g}(g_l, D_{g_k} g_k) \text{ by (A.2)} \\ \mathsf{g}\left(g_l, -\sum_{k=1}^{m} D_{g_k} g_k\right). \end{cases}$$

This ends the proof. □

References

[1] M. Cohen de Lara, A note on the symmetry group and perturbation algebra of a parabolic partial differential operator, J. Math. Phys. 32 (6) (1991) 1444–1449.

[2] M. Cohen de Lara, Geometric and symmetry properties of a nondegenerate diffusion process, Ann. Probab. 23 (4) (1995) 1557–1604.

[3] M. Cohen de Lara, Finite dimensional filters. Part II. Invariance group techniques, SIAM J. Contr. Optim. 35 (3) (1997) 1002–1029.

[4] M. Cohen de Lara, Characterization of a subclass of finite dimensional estimation algebras with maximal rank. Application to filtering, Math. Contr. Sign. Syst. 10 (1997) 237–246.
[5] M. Cohen de Lara, Reduction of the Zakai equation by invariance group techniques, Stochastic Process Appl. 73 (1998) 119–130.
[6] L.V. Ovsjannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.
[7] R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, Tensor Analysis, and Applications, Springer-Verlag, New York, 1988.
[8] H. Spohn, Kinetic equations from hamiltonian dynamics: Markovian limits, Rev. Mod. Phys. 53 (1980) 569–615.
[9] H. Spohn (Ed.), Large Scale Dynamics of Interacting Particles, Springer-Verlag, Berlin, 1991.
[10] Ya.G. Sinaï (Ed.), Dynamical Systems. II. Ergodic Theory with Applications to Dynamical Systems and Statistical Dynamics, Springer-Verlag, Berlin, 1989.
[11] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North Holland, Amsterdam, 1989.
[12] M. Emery, Stochastic Calculus in Manifolds, Springer-Verlag, Berlin, 1989.
[13] L. Arnold, Stochastic Differential Equations. Theory and Applications, Wiley, New York, 1973.
[14] K. Elworthy, Stochastic dynamical systems and their flows, in: A. Press (Ed.), Proceedings of the International Conference on Stochastic Analysis, Northwestern University, Evanston, IL, New York–London, 1978, pp. 79–95.
[15] G.C. Papanicolaou, Asymptotic analysis of transport processes, Bull. Am. Math. Soc. (N.S.) 81 (2) (1975) 330–392.
[16] G. Papanicolaou, Asymptotic analysis of stochastic equations, in: M. Rosenblatt (Ed.), Studies in Probability Theory, vol. 18, The Mathematical Association of America, 1978, pp. 111–179.
[17] H. Kesten, G. Papanicolaou, A limit theorem for turbulent diffusion, Commun. Math. Phys. 65 (1979) 97–128.
[18] H. Kesten, G. Papanicolaou, A limit theorem for stochastic acceleration, Commun. Math. Phys. 78 (1980) 19–63.
[19] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 1, Wiley, New York, 1963.
[20] M. Liao, Symmetry groups of Markov processes, Ann. Probab. 20 (1992) 563–578.